1.7 DIVIDING A POLYNOMIAL BY A POLYNOMIAL

We can illustrate the division statement with the following example.

$$\begin{array}{r} 2 \\ 17\overline{)38} \\ \underline{34} \\ 4 \end{array} \qquad \begin{array}{r} \text{quotient} \\ \text{divisor}\overline{)\text{dividend}} \\ \\ \text{remainder} \end{array}$$

We can check the division using the division statement.

The division statement is

(quotient) × (divisor) + remainder = dividend

From the example, $(2 \times 17) + 4 = 38.$

We use this method to divide a polynomial by a polynomial.

EXAMPLE 1. Divide $(2x^2 - 3x - 1)$ by $(x + 2)$ and write the division statement.

SOLUTION:

Divide the first term of the dividend by the first term of the divisor.

$$\begin{array}{r} 2x \phantom{{}-3x-1} \\ x+2\overline{)2x^2 - 3x - 1} \end{array}$$

Multiply the first term of the quotient by the divisor.

$$\begin{array}{r} 2x \phantom{{}-3x-1} \\ x+2\overline{)2x^2 - 3x - 1} \\ 2x^2 + 4x \phantom{{}-1} \end{array}$$

Subtract the resulting product from the dividend.

$$\begin{array}{r} 2x \phantom{{}-3x-1} \\ x+2\overline{)2x^2 - 3x - 1} \\ \underline{2x^2 + 4x} \phantom{{}-1} \\ -7x - 1 \end{array}$$

Divide the first term of the remainder by the first term of the divisor.

$$\begin{array}{r} 2x - 7 \\ x+2\overline{)2x^2 - 3x - 1} \\ \underline{2x^2 + 4x} \phantom{{}-1} \\ -7x - 1 \end{array}$$

Multiply the second term of the quotient by the divisor.

$$\begin{array}{r} 2x - 7 \\ x+2\overline{)2x^2 - 3x - 1} \\ \underline{2x^2 + 4x} \phantom{{}-1} \\ -7x - 1 \\ -7x - 14 \end{array}$$

Subtract the resulting product from the previous remainder.

$$\begin{array}{r} 2x - 7 \\ x+2\overline{)2x^2 - 3x - 1} \\ \underline{2x^2 + 4x} \phantom{{}-1} \\ -7x - 1 \\ \underline{-7x - 14} \\ 13 \end{array}$$

Stop when the remainder is zero or the degree of the remainder is less than the degree of the divisor.

Division Statement:

(quotient) × (divisor) + remainder = dividend

$(2x - 7) \times (x + 2) + 13 = 2x^2 - 3x - 1$

EXAMPLE 2.
Divide $18x - 19x^2 + 6x^3 - 22$ by $2x - 5$ and state any restrictions on the variable.

SOLUTION:
We first rearrange the terms of the dividend in descending powers of x.

$$(6x^3 - 19x^2 + 18x - 22) \div (2x - 5)$$

$$\begin{array}{r} 3x^2 - 2x + 4 \\ 2x-5\overline{)6x^3 - 19x^2 + 18x - 22} \\ \underline{6x^3 - 15x^2} \phantom{{}+18x-22} \\ -4x^2 + 18x \phantom{{}-22} \\ \underline{-4x^2 + 10x} \phantom{{}-22} \\ 8x - 22 \\ \underline{8x - 20} \\ -2 \end{array}$$

$\therefore 6x^3 - 19x^2 + 18x - 22$

$= (2x - 5)(3x^2 - 2x + 4) - 2$

Since division by 0 is not defined,

$$\begin{aligned} 2x - 5 &\neq 0 \\ 2x &\neq 5 \\ x &\neq \tfrac{5}{2} \end{aligned}$$

EXAMPLE 3.
Divide $(x^4 + x^3y - xy^3 - y^4)$ by $(x^2 - y^2)$.

SOLUTION:
Represent missing terms in the dividend by using zero as the coefficient.

$$\begin{array}{r} x^2 + xy + y^2 \\ x^2 - y^2\overline{)x^4 + x^3y + 0x^2y^2 - xy^3 - y^4} \\ \underline{x^4 \phantom{{}+x^3y} - x^2y^2} \phantom{{}-xy^3-y^4} \\ x^3y + x^2y^2 - xy^3 \phantom{{}-y^4} \\ \underline{x^3y \phantom{{}+x^2y^2} - xy^3} \phantom{{}-y^4} \\ x^2y^2 \phantom{{}-xy^3} - y^4 \\ \underline{x^2y^2 \phantom{{}-xy^3} - y^4} \\ 0 \end{array}$$

$\therefore x^4 + x^3y - xy^3 - y^4$

$= (x^2 - y^2)(x^2 + xy + y^2)$

EXERCISE 1.7

B 1. Divide and state any restrictions on the variables.

(a) $(x^3 - 2x^2 + 2x - 15) \div (x - 3)$
(b) $(x^3 + 3x^2 - 9x - 20) \div (x + 4)$
(c) $(x^3 + 2x^2 - 5x - 7) \div (x + 3)$
(d) $(5w^2 - 4w - 2 + w^3) \div (w - 1)$
(e) $(11x^2 - 22 + 26x + x^3) \div (6 + x)$
(f) $(3t - 6 - 2t^2 + t^3) \div (t - 2)$
(g) $(24 + 6x - 7x^2 + x^3) \div (x - 5)$
(h) $(5x^2 + x^3 - 4x - 20) \div (5 + x)$
(i) $(x^4 + 4x^3 + 2x^2 - 3x + 2) \div (x + 2)$
(j) $(2w - 4w^2 + 2w^4 - 5w^3 + 3) \div (w - 3)$

2. Divide and state any restrictions on the variables.

(a) $(2x^3 + x^2 + x - 1) \div (2x - 1)$
(b) $(21w - 11w^2 + 3w^3 - 7) \div (3w - 2)$
(c) $(2 + 5t - t^2 + 6t^3) \div (1 + 3t)$
(d) $(6z^3 + 13z^2 - 9) \div (2z + 3)$
(e) $(9x^2 - 8 + 4x^3) \div (2 + x)$
(f) $(4x^3 + 5x + 21) \div (2x + 3)$
(g) $(2w - 1 + 9w^3) \div (3w - 2)$
(h) $(10 + 9x + x^3) \div (2 + x)$

C 3. Divide. No divisors are zero.

(a) $(x^4 + x^3 - 13x^2 - 25x - 12) \div (x^2 + 2x + 1)$
(b) $(2w^3 - 4 - 8w - 3w^2 + w^4) \div (w^2 - w - 2)$
(c) $(t^4 - 17t^2 - 36t - 20) \div (t^2 - 3t - 10)$
(d) $(x^3 + x^2y - xy^2 - y^3) \div (x - y)$
(e) $(x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4) \div (x^2 + y^2)$
(f) $(x^3 - 4x^2y + 5xy^2 - 2y^3) \div (x - 2y)$

1.8 SYNTHETIC DIVISION

When dividing by $x - n$ where the coefficient of x is 1, division is simplified by using a process called synthetic division. It is derived from the standard division procedure by working with only the numerical coefficients. The process is illustrated in the following example.

EXAMPLE 1. Divide $-3x^2 - 10x + 2x^3 + 5$ by $x - 3$, $x \neq 3$.

SOLUTION:
We first rearrange the terms of the dividend in descending powers of x.

$$2x^3 - 3x^2 - 10x + 5$$

Copy the coefficients and place the n number (in this case 3) as shown.

$$3 \; \big| \; 2 \quad -3 \quad -10 \quad 5$$

Repeat the first coefficient of the dividend as shown.

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -10 & 5 \\ & \downarrow & & & \\ \hline & 2 & & & \end{array}$$

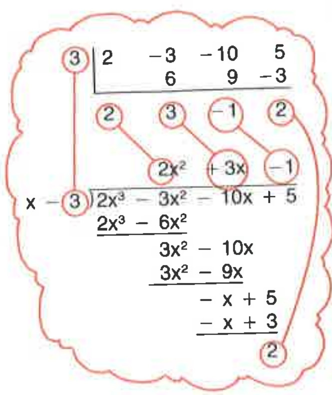
Multiply the number below the line by n, put the product above the line in the second column, and add.

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -10 & 5 \\ & & 6 & & \\ \hline & 2 & 3 & & \end{array}$$

Multiply the last number below the line by n and continue the above process.

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -10 & 5 \\ & & 6 & 9 & -3 \\ \hline & 2 & 3 & -1 & 2 \end{array}$$

When the process is complete, the last number on the row is the remainder and the other numbers are the coefficients of the quotient.



Thus, when $2x^3 - 3x^2 - 10x + 5$ is divided by $x - 3$ the quotient is $2x^2 + 3x - 1$ and the remainder is 2.

EXAMPLE 2. Use synthetic division to divide $x^3 - x + 28$ by $x + 3$ and write the division statement. State any restrictions on the variable.

SOLUTION:
We are dividing by $x + 3$ or $x - (-3)$, so $n = -3$. An x^2 term is missing from the dividend so we use a 0 coefficient as a placeholder.

$$\begin{array}{r|rrrr} -3 & 1 & 0 & -1 & 28 \\ & & -3 & 9 & -24 \\ \hline & 1 & -3 & 8 & 4 \end{array}$$

$\therefore x^3 - x + 28 = (x + 3)(x^2 - 3x + 8) + 4$
In order to avoid division by 0 we stipulate that $x \neq -3$.

The next example illustrates how we use synthetic division to divide by a polynomial of the form $ax + b$. We write $ax + b$ as $a\left(x + \frac{b}{a}\right)$ and divide as before.

EXAMPLE 3. Divide $(2x^3 - 7x^2 - 10x + 26)$ by $(2x - 3)$.

SOLUTION:

$$2x - 3 = 2(x - \tfrac{3}{2})$$

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & -7 & -10 & +26 \\ & & 3 & -6 & -24 \\ \hline & 2 & -4 & -16 & 2 \end{array}$$

$$\begin{aligned} 2x^3 - 7x^2 - 10x + 26 &= \tfrac{1}{2}(2x - 3)(2x^2 - 4x - 16) + 2 \\ &= (2x - 3)(x^2 - 2x - 8) + 2 \end{aligned}$$

Why must we multiply by $\frac{1}{2}$?

EXERCISE 1.8

B 1. Divide using synthetic division and state restrictions.
(a) $(x^3 + 2x^2 - 8x - 2) \div (x - 1)$
(b) $(x^3 - 13x^2 + 20x + 20) \div (x - 5)$
(c) $(x^3 + 10x^2 + 29x + 20) \div (x + 4)$
(d) $(y^2 + y^3 - 20) \div (y + 2)$
(e) $(t^3 - 8t^2 + 19t - 12) \div (t - 3)$

2. Divide using synthetic division and state restrictions.
(a) $(2x^3 - 4x - 4x^2 + 2) \div (x + 4)$
(b) $(2x^3 - 13x + 12) \div (x + 3)$
(c) $(3t^4 - 7 - 14t^2 + 10t + t^3) \div (t + 3)$
(d) $(3x^4 - 12x^3 - 20x^2 - 30x + 2) \div (x - 5)$
(e) $(20 - 14y^3 - y + 4y^4) \div (y - 3)$
(f) $(x^3 - 2x^2 - 75) \div (x - 5)$
(g) $(t^3 - 4t^2 + t + 6) \div (t - 2)$

3. Divide using synthetic division and state restrictions.
(a) $(6x^2 - 11x + 7) \div (3x - 4)$
(b) $(6t^3 - 5t^2 - 13t + 13) \div (2t + 3)$
(c) $(15y^3 - 18y - y^2 + 8) \div (3y - 2)$
(d) $(2y^3 - 9y^2 + 11y - 3) \div (2y - 3)$
(e) $(12 + 8t^3 - 22t^2 - 5t) \div (4t + 3)$
(f) $(4y^3 - 12y^2 - 37y - 14) \div (2y + 1)$
(g) $(5x^2 - 13x + 10 + 6x^3) \div (3x - 5)$

1.9 THE REMAINDER THEOREM

Given the polynomial $P(x) = x^3 - x^2 - 7x + 9$, we divide P(x) by $(x - 3)$ and then find P(3).

$$\begin{array}{r}
x^2 + 2x - 1 \\
x - 3\overline{)x^3 - x^2 - 7x + 9} \\
\underline{x^3 - 3x^2 } \\
2x^2 - 7x \\
\underline{2x^2 - 6x } \\
-x + 9 \\
\underline{-x + 3} \\
6
\end{array}$$

$$\begin{aligned}
P(x) &= x^3 - x^2 - 7x + 9 \\
P(3) &= (3)^3 - (3)^2 - 7(3) + 9 \\
&= 27 - 9 - 21 + 9 \\
&= 6
\end{aligned}$$

When P(x) is divided by $(x - 3)$ the remainder is P(3). This leads to a statement of the Remainder Theorem.

> **Remainder Theorem**
>
> When a polynomial P(x) is divided by $(x - b)$, and the remainder contains no term in x, then the remainder is P(b).

$$\begin{array}{r} Q(x) \\ x - b\overline{)P(x)} \\ R \end{array}$$

Proof:
The following division statement is true for all values of x.

$$P(x) = (x - b) \times Q(x) + R$$

Q(x) represents a polynomial in x and R is a constant.
Substituting b for x we have

$$\begin{aligned}
P(b) &= (b - b) \times Q(b) + R \\
P(b) &= 0 \times Q(b) + R \\
P(b) &= R
\end{aligned}$$

In general,

$$(ax - b) = a\left(x - \frac{b}{a}\right)$$

> When a polynomial P(x) is divided by $(ax - b)$, and the remainder contains no term in x, then the remainder is $P\left(\frac{b}{a}\right)$.

EXAMPLE 1. Use the Remainder Theorem to determine the remainder when $2x^3 - 4x^2 + 3x - 6$ is divided by $x + 2$.

SOLUTION:

$$\begin{aligned}
P(x) &= 2x^3 - 4x^2 + 3x - 6 \\
P(-2) &= 2(-2)^3 - 4(-2)^2 + 3(-2) - 6 \\
&= -16 - 16 - 6 - 6 \\
&= -44
\end{aligned}$$

The remainder is -44.

remainder when $2x^3 - 13x^2 + 19x + 7$ is divided by $2x - 3$.

SOLUTION:

$$\begin{aligned}
P(x) &= 2x^3 - 13x^2 + 19x + 7 \\
P(\tfrac{3}{2}) &= 2(\tfrac{3}{2})^3 - 13(\tfrac{3}{2})^2 + 19(\tfrac{3}{2}) + 7 \\
&= \tfrac{27}{4} - \tfrac{117}{4} + \tfrac{57}{2} + 7 \\
&= \tfrac{27}{4} - \tfrac{117}{4} + \tfrac{114}{4} + \tfrac{28}{4} \\
&= \tfrac{52}{4} \\
&= 13
\end{aligned}$$

The remainder is 13.

EXERCISE 1.9

B

1. For $P(x) = x^3 + x^2 + x + 4$, find.
(a) P(2) (b) P(1)
(c) P(0) (d) P(−2)

2. For $f(x) = x^3 - 2x^2 - 2x - 3$, find.
(a) f(3) (b) f(−1)
(c) f(−3) (d) f(5)

3. For $g(x) = 2x^2 - 4x + 5$, find.
(a) g(−3) (b) $g(\frac{1}{2})$
(c) $g(-\frac{1}{2})$ (d) $g(\frac{3}{2})$

4. Use the Remainder Theorem to determine the remainder for each division.
(a) $(x^3 + 2x^2 + 3x + 7) \div (x - 1)$
(b) $(m^3 - m^2 + 7m - 4) \div (m + 1)$
(c) $(x^3 - 8x^2 + 17x - 6) \div (x - 3)$
(d) $(n^4 - 3n^2 + 7) \div (n + 3)$
(e) $(1 - 2x - 3x^2 + x^3) \div (x - 1)$
(f) $(x^3 - 27) \div (x - 3)$
(g) $(2x^3 - 3x - 4) \div (x + 2)$
(h) $(3m^3 - m^2 - m - 2) \div (m + 1)$

C

5. Find the remainder for each division.
(a) $(2x^2 + 5x - 4) \div (2x - 1)$
(b) $(2x^2 - 5x + 2) \div (2x + 1)$
(c) $(5m^3 + m^2 - 5m - 3) \div (2m + 3)$
(d) $(3x^3 - x^2 - 12x + 7) \div (3x - 1)$
(e) $(2m^3 + 3m^2 - 7m - 5) \div (2m + 5)$

CALCULATOR MATH

You can evaluate a polynomial on a calculator using the "multiply and add" cycle of synthetic division.

To evaluate $2x^3 + 3x^2 - 4x + 7$ for $x = 5$

$$\begin{array}{r|rrrr}
5 & 2 & 3 & -4 & 7 \\
 & & 10 & 65 & 305 \\
\hline
 & 2 & 13 & 61 & 312
\end{array}$$

2 × 5 = + 3 = × 5 = − 4 = × 5 = + 7 =

EXERCISE

1. Evaluate each polynomial.
(a) $3x^3 - 2x^2 + x - 9$ for $x = 2$
(b) $4x^3 + 5x^2 - 3x - 11$ for $x = 6$
(c) $2x^4 - 5x^3 + 7x^2 - 3x + 7$ for $x = -2$
(d) $5x^4 + 3x^3 - x^2 + 7x - 3$ for $x = -4$

1.10 THE FACTOR THEOREM

We now use a corollary of the Remainder Theorem, the Factor Theorem, to factor polynomials of the third degree and higher.

EXAMPLE 1. Find the remainder when $x^3 - 7x^2 + 9x + 2$ is divided by $x - 2$ and write the division statement.

SOLUTION:

$$\begin{aligned} P(x) &= x^3 - 7x^2 + 9x + 2 \\ P(2) &= (2)^3 - 7(2)^2 + 9(2) + 2 \\ &= 8 - 28 + 18 + 2 \\ &= 0 \end{aligned}$$

$$\begin{array}{r} x^2 - 5x - 1 \\ x - 2\overline{)x^3 - 7x^2 + 9x + 2} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ -5x^2 + 9x \qquad \\ \underline{-5x^2 + 10x} \qquad \\ -x + 2 \\ \underline{-x + 2} \\ 0 \end{array} \quad \text{or} \quad \begin{array}{r|rrrr} 2 & 1 & -7 & +9 & +2 \\ & & 2 & -10 & -2 \\ \hline & 1 & -5 & -1 & 0 \end{array}$$

$\therefore x^3 - 7x^2 + 9x + 2 = (x + 2)(x^2 - 5x - 1)$

Since division gives zero as a remainder, both $x - 2$ and $x^2 - 5x - 1$ are factors of $x^3 - 7x^2 + 9x + 2$. This illustrates the factor theorem.

> **Factor Theorem**
>
> A polynomial $P(x)$ has $x - b$ as a factor if and only if $P(b) = 0$.

If a polynomial $P(x)$ has $x - b$ as a factor, then

$$P(x) = (x - b) \times Q(x)$$

Substituting b for x we have

$$\begin{aligned} P(b) &= (b - b) \times Q(b) \\ P(b) &= 0 \end{aligned}$$

Conversely, if $P(b) = 0$, then by the Remainder Theorem when $P(x)$ is divided by $x - b$ the remainder is 0, that is, $x - b$ is a factor of $P(x)$.

EXAMPLE 2. Factor. $4x^3 + 16x^2 + 9x - 9$

SOLUTION:

$$\begin{aligned} P(x) &= 4x^3 + 16x^2 + 9x - 9 \\ P(1) &= 4(1)^3 + 16(1)^2 + 9(1) - 9 \\ &= 20 \end{aligned}$$

In this example we only substitute numbers that are factors of 9. Why?

$$\begin{aligned} &= -6 \\ P(3) &= 4(3)^3 + 16(3)^2 + 9(3) - 9 \\ &= 270 \\ P(-3) &= 4(-3)^3 + 16(-3)^2 + 9(-3) - 9 \\ &= 0 \end{aligned}$$

$\therefore x + 3$ is a factor.

Another factor is found by division.

$$\begin{array}{r|rrrr} -3 & 4 & 16 & 9 & -9 \\ & & -12 & -12 & 9 \\ \hline & 4 & 4 & -3 & 0 \end{array}$$

$$\begin{array}{r} 4x^2 + 4x - 3 \\ x + 3\overline{)4x^3 + 16x^2 + 9x - 9} \\ \underline{4x^3 + 12x^2} \qquad\qquad \\ 4x^2 + 9x \qquad \\ \underline{4x^2 + 12x} \qquad \\ -3x - 9 \\ \underline{-3x - 9} \\ 0 \end{array}$$

$\therefore 4x^3 + 16x^2 + 9x - 9 = (x + 3)(4x^2 + 4x - 3)$

$4x^2 + 4x - 3$ may be factored.

$\therefore 4x^3 + 16x^2 + 9x - 9 = (x + 3)(2x + 3)(2x - 1)$

EXAMPLE 3. Find k so that $x^3 - 4x^2 - 2x + k$ has $x - 3$ as a factor.

SOLUTION:

Let $P(x) = x^3 - 4x^2 - 2x + k$. If $x - 3$ is a factor, $P(3) = 0$.

$$\begin{aligned} P(x) &= x^3 - 4x^2 - 2x + k \\ P(3) &= (3)^3 - 4(3)^2 - 2(3) + k = 0 \\ 27 &- 36 - 6 + k = 0 \\ -15 &+ k = 0 \\ k &= 15 \end{aligned}$$

EXERCISE 1.10

A

1. Determine which of the polynomials have $x - 1$ as a factor.
   (a) $x^3 + x^2 - x - 1$
   (b) $2x^3 - x^2 - 3x - 1$
   (c) $x^4 - 3x^3 + 2x^2 - x + 1$
   (d) $3x^3 - x - 3$
   (e) $4x^4 - 2x^3 + 3x^2 - 2x + 1$
   (f) $x^3 - 3x^2 + 4x - 2$
   (g) $2x^3 + 4x^2 - 5x - 1$
   (h) $x^3 - x^2 - x - 1$

2. Is $(x + 1)$ a factor of $x^{100} - 1$? Is $(x - 1)$? Explain.

3. Is $(x + 1)$ a factor of $x^{99} + 1$? Is $(x - 1)$? Explain.

B

4. Use the factor theorem to show that
   (a) $(x + 1)$ is a factor of $x^3 + 2x^2 + 2x + 1$.
   (b) $(x - 2)$ is a factor of $x^3 - 3x^2 + 4x - 4$.
   (c) $(x - 3)$ is a factor of $x^3 - 3x^2 + x - 3$.
   (d) $(x + 3)$ is a factor of $x^3 + 7x^2 + 17x + 15$.
   (e) $(x + 2)$ is a factor of $2x^3 + 4x^2 - 3x - 6$.
   (f) $(x + 5)$ is a factor of $x^4 + 5x^3 + 2x^2 + 7x - 15$.
   (g) $(x - 4)$ is a factor of $2x^4 - 11x^3 + 12x^2 + x - 4$.

6. Factor the following over the integers.

(a) $x^3 - 6x^2 + 11x - 6$
(b) $x^3 + 8x^2 + 19x + 12$
(c) $t^3 - 2t^2 - 9t + 18$
(d) $m^3 + 4m^2 + 2m - 3$
(e) $x^3 + x^2 - 22x - 40$
(f) $x^3 + x^2 - 16x - 16$
(g) $w^3 - 2w^2 - 6w - 8$
(h) $n^3 + 6n^2 - 7n - 60$
(i) $x^3 - 27$
(j) $x^3 - 27x + 10$

7. Factor the following.

(a) $2x^3 - 9x^2 + 10x - 3$
(b) $3x^3 - 2x^2 - 12x + 8$
(c) $2x^3 - 3x^2 + 3x - 10$
(d) $4m^3 - 7m - 3$
(e) $5t^3 - 23t^2 + 20t + 12$
(f) $4y^3 + 13y^2 + 4y + 3$
(g) $6x^3 - 11x^2 - 26x + 15$
(h) $2x^3 - 4x^2 + 11x - 22$
(i) $2x^4 + x^3 - 26x^2 - 37x - 12$
(j) $x^5 - 3x^4 - 10x^3 + 3x^2 - 9x - 30$

C 8. Find k so that $x^3 - 2x^2 + 3x + k$ has $(x - 1)$ as a factor.

9. Find k so that $x^3 + 5x^2 + kx + 6$ has $(x + 2)$ as a factor.

10. Find k so that $km^3 - 10m^2 + 2m + 3$ has $(m - 3)$ as a factor.

11. Find k so that
$3x^3 - 2kx^2 + (k - 1)x + 10$
has $x + 2$ as a factor.

12. Find k so that
$2x^3 + (k + 1)x^2 + 6kx + 11$
has $x - 1$ as a factor.

13. Find the value of k so that when $x^2 + 8x + k$ is divided by $x - 2$ the remainder is 3.

14. Find the value of k so that when $x^3 + 5x^2 + 6x + 11$ is divided by $x + k$ the remainder is 3.

MICRO MATH

We can write a BASIC program to factor third degree polynomials using synthetic division. Let us take a general polynomial.

$$(Ax^3 + Bx^2 + Cx + D) \div (x - x_1)$$

x_1	A	B	C	D
		A_1x	B_1x	C_1x
	A_1	$B_1 = B + A_1x$	$C_1 = C + B_1x$	$D_1 = D + C_1x$

Since D_1 is the remainder, the polynomial is divisible by $(x - x_1)$ when $D_1 = 0$. These ideas are applied in the program listed below.

In the BASIC computer language, subscripted variables such as x_1 are used by writing x1.

This program uses synthetic division to factor a polynomial of degree 3.

```
NEW
100 PRINT"FACTORING OF A"
110 PRINT"POLYNOMIAL OF DEGREE 3."
120 PRINT"AX↑3 + BX↑2 + CX + D"
130 PRINT"ENTER THE COEFFICIENTS AND"
140 PRINT"CONSTANT TERM SEPARATED BY"
150 PRINT"COMMAS."
160 INPUT A,B,C,D
170 PRINT
180 PRINT"ENTER A TRIAL VALUE FOR X>"
190 INPUT X
200 A1=A
210 B1=B+A1*X
220 C1=C+B1*X
230 D1=D+C1*X
240 IF D1<>0 THEN 170
250 PRINT
260 X1=-X
270 PRINT"THE FACTORED POLYNOMIAL:"
280 PRINT
290 PRINT "(X + ";X1;")(";A1;"X↑2 + "
        ;B1;"X + ";C1;")"
300 END
RUN
```


The polynomial $x^3 + 8$ is the sum of two cubes.

The polynomial $x^3 - 27$ is the difference of two cubes.

There is a pattern for factoring each of these polynomials.

EXAMPLE. Factor. $x^3 + 8$

SOLUTION:

$$\begin{aligned} P(x) &= x^3 + 8 \\ P(-2) &= (-2)^3 + 8 \\ &= 0 \end{aligned}$$

$(x + 2)$ is a factor.

Another factor is found by division.

−2	1	0	0	8
		−2	4	−8
	1	−2	4	0

$\therefore x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

In general,

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

EXERCISE 1.11

B 1. Use the distributive property to show that
(a) $(x + y)(x^2 - xy + y^2) = x^3 + y^3$.
(b) $(x - y)(x^2 + xy + y^2) = x^3 - y^3$.

2. Factor.

(a) $x^3 - 27$
(b) $x^3 + 64$
(c) $m^3 - 8$
(d) $y^3 + 125$
(e) $8x^3 - 1$
(f) $64m^3 - 1$
(g) $27x^3 + 8y^3$
(h) $x^6 + y^9$
(i) $s^6 - 27t^9$
(j) $1000x^{12} + y^{15}$
(k) $a^3b^3 + 27c^3$
(l) $x^3y^6 - 9z^9$
(m) $\frac{1}{8}x^3 - 1$
(n) $m^6 - \frac{1}{27}n^3$

C 3. Show that $(x - a)$ is a factor of $x^n - a^n$.

4. (a) If n is even, is $(x + a)$ a factor of $x^n + a^n$?
(b) If n is odd, is $(x + a)$ a factor of $x^n + a^n$?

5. Is $(x - a)$ a factor of $x^2(a - b) + a^2(b - x) + b^2(x - a)$?

6. Use the Factor Theorem to show that $(x - c)$ is a factor of $(x - b)^2 - (b - c)^2 + (c - x)^2$.

7. Factor $x - 1$ as a difference of cubes.

MIND BENDER

It takes 24 h for Ed and Alex to do a job together. How long will it take each to do the job alone if Ed works $\frac{2}{3}$ as fast as Alex?

1. (a) $(a + b)(3t + 7)$ (b) $(x - 1)(5 + 2x)$ (c) $(x - 5)(3x - 7)$
(d) $(x + y)(4m + 1)$ (e) $(m - n)(2a - 1)$ (f) $(2x - y)(4x - 3)$
(g) $(x + 3)(x + 1)$ (h) $(x^2 - x - 1)(m + 3)$ (i) $(x - y)(7q - 1)$
2. (a) $(a + b)(x + y)$ (b) $(x + y)(m + 2)$ (c) $(1 - y)(1 + y^2)$
(d) $(x + y)(m - n)$ (e) $(a - b)(x - 3)$ (f) $(x - 1)(cx - dx - d)$
3. (a) $(x + 1)(x - 1)(x - 2)$ (b) $(4m - n)(x - 2y)$ (c) $(y - 2x)(3 - 5x)$
(d) $(5s - 4)(3s - t)$ (e) $(a - 2b)(a - 3c)$ (f) $(3a + 4d)(b - 5c)$
(g) $(4y + 3)(y + 2x)$ (h) $(y - x)(x^2 - y)$
4. (a) $(a^2 - 4)(b^2 - 7b + 13)$ (b) $(r^2 - 9)(x^2 - 3x + 2)$ (c) $(a - b)(x^2 + x + 1)$

EXERCISE 1.5

1. (a) $(x + 3)(x + 4)$ (b) $(x + 2)(x + 5)$ (c) $(w - 3)(w - 5)$ (d) $(x + 2)(x - 4)$
(e) $(x + 5)(x - 2)$ (f) $(m + 1)(m - 7)$ (g) $(t - 4)(t - 5)$ (h) $(x + 4)(x - 6)$
(i) $(x + 5)(x - 3)$ (j) $(t + 3)(t - 4)$ (k) $(w + 5)(w - 9)$ (l) $(r + 5)(r + 7)$
(m) $(x + 4)(x + 7)$ (n) $(w - 4)(w - 10)$ (o) $(t + 4)(t - 5)$ (p) $(x + 11)(x - 8)$
(q) $(r + 6)(r - 4)$ (r) $(y - 2)(y - 5)$ (s) $(x + 10)(x - 3)$ (t) $(s + 3)(s - 7)$
(u) $(x + 9)(x - 3)$ (v) $(t - 1)(t - 2)$ (w) $(x + 5)(x - 8)$ (x) $(w + 10)(w - 7)$
(y) $(m + 7)(m + 9)$ (z) $(n - 2)(n - 8)$
2. (a) $-4, 3$ (b) $2, 5$ (c) not possible (d) $-5, 10$
(e) $-9, 4$ (f) not possible (g) $4, 11$ (h) $-10, -2$
(i) $-1, -1$ (j) $-3, 6$ (k) $-6, 6$ (l) $-4, -4$
3. (a) $(x + 3)(2x + 1)$ (b) $(2x - 5)(x - 1)$ (c) $(3w + 4)(w - 5)$ (d) $(3y - 1)(2y + 1)$
(e) $(2w - 3)(3w + 1)$ (f) $(5w + 2)(2w - 1)$ (g) not possible (h) $(x - 1)(2x - 1)$
(i) $(2w + 5)(w + 2)$ (j) $(2t + 3)(t + 2)$ (k) not possible (l) $(3w - 5)(w + 4)$
(m) not possible (n) $(5x - 2)(2x + 1)$ (o) $(4x - 3)(2x + 3)$ (p) $(4t - 5)(t - 2)$
4. (a) $2(x - 1)(2x - 3)$ (b) $3(3m + 5)(m + 2)$ (c) not possible (d) $(8y + 7)(2y - 3)$
(e) $(3y - 2)(4y - 3)$ (f) $2(2x + 3)(5x + 4)$ (g) not possible (h) not possible
(i) $2(2x + 3)(3x - 5)$ (j) $(2t + 7)(3t - 4)$

EXERCISE 1.6

1. (a) $(x - 4)^2$ (b) $(y + 5)^2$ (c) $(x - 4)(x + 4)$
(d) $(m + 6)^2$ (e) $(t - 7)(t + 7)$ (f) $(w - 7)^2$
(g) $(x - 10)^2$ (h) $(s - 1)(s + 1)$ (i) $(y + 1)^2$
2. (a) $(2x - 3)(2x + 3)$ (b) $(6y - 7)(6y + 7)$ (c) $(10x - 9)(10x + 9)$
(d) $(1 - 4y)(1 + 4y)$ (e) $(11y - 2)(11y + 2)$ (f) $(3m - 1)(3m + 1)$
3. (a) $(4x - 7y)(4x + 7y)$ (b) $(2m + 3n)^2$ (c) $(6s - 5t)^2$
(d) $(9p + 8q)^2$ (e) $(3d - 5y)(3d + 5y)$ (f) $(4x - 11y)^2$
(g) $(5a + 3b)^2$ (h) $(7s - 4t)^2$ (i) $(10m - 11n)(10m + 11n)$
4. (a) $(7x^2y - 2z)(7x^2y + 2z)$ (b) $(5mn + 4t)^2$ (c) $(6x^3 - 5y^2)(6x^3 + 5y^2)$
(d) $(2x^3y^2 - 7)^2$ (e) $(4m^3 - 5n)^2$ (f) $(6s^2 + 5t)^2$
(g) $(0.5x - 0.8y)(0.5x + 0.8y)$ (h) $(1.2s + 2.5t^2)^2$
5. (a) $(x - y - 4)(x - y + 4)$ (b) $(s + 3t - 3)(s + 3t + 3)$
(c) $(a + 2b - 12)(a + 2b + 12)$ (d) $(3x - 2y - 5)(3x - 2y + 5)$
6. (a) $(7 - 2y + w)(7 + 2y - w)$ (b) $(1 - x + y)(1 + x - y)$
(c) $(x + y + a + b)(x + y - a - b)$ (d) $(3m + n - 2s + 5t)(3m + n + 2s - 5t)$
7. (a) $(x + 3y - 6)(x + 3y + 6)$ (b) $(s - 4 - \sqrt{t})(s - 4 + \sqrt{t})$
(c) $(p + q - 5)(p + q + 5)$ (d) $(x - y - 7)(x + y + 7)$
(e) $(a + b - c + 3)(a + b + c - 3)$ (f) $(m + n - s - t)(m + n + s + t)$
(g) $(5y - 3 - 2c - d)(5y - 3 + 2c + d)$
8. (a) $-(3y - 2)^2$ (b) $2ab(a - 1)^2$ (c) $3(3 - x)(x + 3)$
(d) $-x(2y + 1)^2$ (e) $2x(x - 5y)(x + 5y)$ (f) $4st(3s + 5t)^2$
(g) $-p(p - 3q)^2$ (h) $(4\sqrt{2} - 3a)(3a + 4\sqrt{2})$
9. (a) 161 (b) 267 (c) 1760 (d) 8760
10. (a) $(x^{2n} - y^{3n})(x^{2n} + y^{3n})$ (b) $(3x^{3n} - 2y^{2n})$ (c) $(4x^{2n+1} + 3y^{4n})^2$
11. (a) 2601 (b) 2401 (c) 1024 (d) 3364

1. (a) $x^2 + x + 5, x \neq 3$ (b) $x^2 - x - 5, x \neq -4$
(c) $x^2 - x - 2$, R -1, $x \neq -3$ (d) $w^2 + 6w + 2, w \neq 1$
(e) $x^2 + 5x - 4$, R 2, $x \neq -6$ (f) $t^2 + 3, t \neq 2$
(g) $x^2 - 2x - 4$, R 4, $x \neq 5$ (h) $x^2 - 4, x \neq -5$
(i) $x^3 + 2x^2 - 2x + 1, x \neq -2$ (j) $2w^3 + w^2 - w - 1, w \neq 3$
2. (a) $x^2 + x + 1, x \neq \frac{1}{2}$ (b) $w^2 - 3w + 5$, R 3, $w \neq \frac{2}{3}$
(c) $2t^2 - t + 2, t \neq -\frac{1}{3}$ (d) $3z^2 + 2z - 3, z \neq -\frac{3}{2}$
(e) $4x^2 + x - 2$, R -4, $x \neq -2$ (f) $2x^2 - 3x + 7, x \neq -\frac{3}{2}$
(g) $3w^2 + 2w + 2$, R 3, $w \neq \frac{2}{3}$ (h) $x^2 - 2x + 13$, R -16, $x \neq -2$
3. (a) $x^2 - x - 12$ (b) $w^2 + 3w + 2$ (c) $t^2 + 3t + 2$
(d) $(x + y)^2$ (e) $(x - y)^2$ (f) $(x - y)^2$

EXERCISE 1.8

1. (a) $x^2 + 3x - 5$, R -7, $x \neq 1$ (b) $x^2 - 8x - 20$, R -80, $x \neq 5$
(c) $x^2 + 6x + 5, x \neq -4$ (d) $y^2 - y + 2$, R -24, $y \neq -2$
(e) $t^2 - 5t + 4, t \neq 3$
2. (a) $2x^2 - 12x + 44$, R -174, $x \neq -4$ (b) $2x^2 - 6x + 5$, R -3, $x \neq -3$
(c) $3t^3 - 8t^2 + 10t - 20$, R 53, $t \neq -3$ (d) $3x^3 + 3x^2 - 5x - 55$, R -273, $x \neq 5$
(e) $4y^3 - 2y^2 - 6y - 19$, R -37, $y \neq 3$ (f) $x^2 + 3x + 15, x \neq 5$
(g) $t^2 - 2t - 3, t \neq 2$
3. (a) $2x - 1$, R 3, $x \neq \frac{4}{3}$ (b) $3t^2 - 7t + 4$, R 1, $t \neq -\frac{3}{2}$
(c) $5y^2 + 3y - 4, y \neq \frac{2}{3}$ (d) $y^2 - 3y + 1, y \neq \frac{3}{2}$
(e) $2t^2 - 7t + 4, t \neq -\frac{3}{4}$ (f) $2y^2 - 7y - 15$, R 1, $y \neq -\frac{1}{2}$
(g) $2x^2 + 5x + 4$, R 30, $x \neq \frac{5}{3}$

EXERCISE 1.9

1. (a) 18 (b) 7 (c) 4 (d) -2
2. (a) 0 (b) -4 (c) -42 (d) 62
3. (a) 35 (b) $\frac{7}{2}$ (c) $\frac{15}{2}$ (d) $\frac{7}{2}$
4. (a) 13 (b) -13 (c) 0 (d) 61
(e) -3 (f) 0 (g) -14 (h) -5
5. (a) -1 (b) 5 (c) $-\frac{81}{8}$ (d) 3 (e) 0

EXERCISE 1.10

1. (a), (c), (f), (g) 2. yes, yes 3. yes, no 5. no
6. (a) $(x - 1)(x - 2)(x - 3)$ (b) $(x + 1)(x + 3)(x + 4)$ (c) $(x - 2)(x - 3)(x + 3)$
(d) $(m + 3)(m^2 + m - 1)$ (e) $(x - 5)(x + 2)(x + 4)$ (f) $(x + 1)(x + 4)(x - 4)$
(g) $(w - 4)(w^2 + 2w + 2)$ (h) $(n - 3)(n + 4)(n + 5)$ (i) $(x - 3)(x^2 + 3x + 9)$
(j) $(x - 5)(x^2 + 5x - 2)$
7. (a) $(x - 1)(x - 3)(2x - 1)$ (b) $(x + 2)(x - 2)(3x - 2)$
(c) $(x - 2)(2x^2 + x + 5)$ (d) $(m + 1)(2m - 3)(2m + 1)$
(e) $(t - 2)(t - 3)(5t + 2)$ (f) $(y + 3)(4y^2 + y + 1)$
(g) $(x - 3)(3x + 5)(2x - 1)$ (h) $(x - 2)(2x^2 + 11)$
(i) $(x + 1)(x + 3)(x - 4)(2x + 1)$ (j) $(x + 2)(x - 5)(x^3 + 3)$
8. -2 9. 9 10. 3 11. $-\frac{6}{5}$ 12. -2
13. -17 14. 4 16. $k = -3$ or 1